

MAS212 Linear Algebra I 2007-2008

Coursework 10

Please hand in your solutions of the **starred** exercises by **2pm on Friday 14 December** using the [blue Linear Algebra I Collection Box](#) on the Ground Floor.

Don't forget to put your **name** and **student number** on your solutions, and to **staple** them.

Exercise* 1. (2 marks) Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. Show the following:

- (a) Q is invertible and $Q^{-1} = Q^T$. [Hint: use Corollary 2.42]
- (b) If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then $\{Q\mathbf{v}_1, \dots, Q\mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. [Hint: use Exercise 7 on Coursework 9]

Exercise* 2. (2 marks) Let $\mathbf{y} = (-1, 4, 3)^T \in \mathbb{R}^3$ and let S be the subspace of $\mathbb{R}^3$ spanned by the orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = (1, 1, 1)^T$, and $\mathbf{u}_2 = (-1, 3, -2)^T$. Write $\mathbf{y}$ as a sum of a vector in S and a vector orthogonal to S .

Exercise* 3. (3 marks) Let S be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and let $\{\mathbf{w}_1, \dots, \mathbf{w}_q\}$ be an orthogonal basis for $S^\perp$.

- (a) Explain why $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1, \dots, \mathbf{w}_q\}$ is an orthogonal set that spans $\mathbb{R}^n$.
- (b) Use (a) to show that $\dim S + \dim S^\perp = n$.

Exercise* 4. (4 marks) Let S be the span of the three linearly independent vectors

$$\mathbf{x}_1 = (1, 1, 0, -1)^T, \quad \mathbf{x}_2 = (-1, -2, 1, 3)^T, \quad \mathbf{x}_3 = (0, -2, 2, 1)^T.$$

Use the Gram Schmidt process to determine an orthogonal basis for S and find the closest point to $\mathbf{y} = (-1, 1, 1, -3)^T$ in S .

Exercise* 5. (3 marks) Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}.$$

Write down the corresponding normal equations and determine the set of least squares solutions.

Exercise 6. Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{x} \in \mathbb{R}^n$.

- (a) Show that if $A\mathbf{x} = \mathbf{0}$ then $A^T A\mathbf{x} = \mathbf{0}$.
- (b) Suppose that $A^T A\mathbf{x} = \mathbf{0}$. Explain why $\mathbf{x}^T A^T A\mathbf{x} = 0$, and use this to show that $A\mathbf{x} = \mathbf{0}$.
- (c) Use (a) and (b) to deduce that $N(A^T A) = N(A)$.
- (d) Use (c) to show that the columns of A are linearly independent if and only if $A^T A$ is invertible.

Exercise* 7. (6 marks) Find the eigenvalues and corresponding eigenspaces for the following matrices:

$$(a) \quad \begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}, \quad (b) \quad \begin{pmatrix} -4 & 0 & 3 \\ -12 & 2 & 6 \\ -6 & 0 & 5 \end{pmatrix}.$$